Two-Dimensional Motion

Projectiles, Airtime, and Range

Calculus & Vector Background[1]

Suppose a projectile is launched with an initial velocity of $\vec{v}_0 = v_{0x}\hat{i} + v_{0y}\hat{j}$ meters per second at an angle of θ° from the horizontal, from an initial position of (x_0, y_0) meters.

The horizontal displacement of a projectile, with no horizontal acceleration, *t* seconds after launch is

$$x(t) = v_{0x}t + x_0 \text{ meters.}$$

The vertical displacement of a projectile is

$$y(t) = -\frac{1}{2}|g|\,t^2 + v_{0y}\,t + y_0 \text{ meters.}$$

where the acceleration due to gravity is $g = -9.81$ meters per second squared.

Formulas and Computations

For convenience, we will set our reference frame so that $x(0) = x_0 = 0.00$ meters.

Initial Horizontal and Vertical Velocity

Using what we know about the magnitude of a vector, the initial *speed* of the projectile is often much easier to measure experimentally than its horizontal and vertical components. We know that the magnitude of the initial velocity vector is

$$||\vec{v}_0|| = \sqrt{(v_{0x})^2 + (v_{0y})^2} \text{ meters per second.}$$

[1] Note: Throughout this experiment, we will ignore any resistance (force) that is the result of air flow; that is, we will ignore *air resistance*. This is a reasonable thing to do since we will be covering short distances with a presumably spherical ball.

Using this formula and trigonometry, we can rewrite

$$v_{0x} = ||\vec{v}_0||\ \cos(\theta) \text{ meters per second}$$

$$v_{0y} = ||\vec{v}_0||\ \sin(\theta) \text{ meters per second}$$

to represent the initial horizontal and vertical velocity of the projectile.

While using these versions of initial horizontal and vertical velocity make the equations look more complex, they do provide for simpler computation later.

Airtime

The *airtime* of a launched projectile is found by determining the time, in seconds, at which $y(t) = 0.00$ meters. The equation will have two real solutions; we need to find the positive solution:

$$-\frac{1}{2}|g|t^2 + v_{0y}t + y_0 = 0$$

$$t = \frac{-v_{0y} \pm \sqrt{(v_{0y})^2 - 4\left(-\frac{1}{2}|g|\right)y_0}}{2\left(-\frac{1}{2}|g|\right)}$$

$$t = \frac{-v_{0y} \pm \sqrt{(v_{0y})^2 + 2|g|y_0}}{-|g|}$$

$$t = \frac{v_{0y} \mp \sqrt{(v_{0y})^2 + 2|g|y_0}}{|g|}$$

Thus, the *airtime* is

$$t = \frac{v_{0y} + \sqrt{(v_{0y})^2 + 2|g|y_0}}{|g|} \text{ seconds.}$$

Substituting the expression for initial vertical velocity, we have

$$t_{air} = \frac{||\vec{v}_0||\sin(\theta) + \sqrt{||\vec{v}_0||^2\sin^2(\theta) + 2|g|y_0}}{|g|} \text{ seconds.}$$

Note that when $y_0 = 0.00$ meters, this equation simplifies to

$$t_{air} = \frac{2\,||\vec{v}_0||\sin(\theta)}{|g|} \text{ seconds.}$$

Further, note that when the projectile is launched at $\theta = 0.00°$, the equation for airtime becomes

$$t_{air} = \sqrt{\frac{2y_0}{|g|}} \text{ seconds}$$

because $\sin(0.00) = 0$.

Range

The horizontal *range* of the projectile is the horizontal displacement at time t_{air}:

$$x(t_{air}) = v_{0x}\, t$$

$$x(t_{air}) = ||\vec{v}_0||\, \cos(\theta)\left(\frac{||\vec{v}_0||\sin(\theta) + \sqrt{||\vec{v}_0||^2 \sin^2(\theta) + 2|g|y_0}}{|g|}\right) \text{ meters.}$$

When $y_0 = 0.00$ meters, this equation simplifies to

$$x(t_{air}) = ||\vec{v}_0||\, \cos(\theta)\left(\frac{2||\vec{v}_0||\sin(\theta)}{|g|}\right)$$

$$x(t_{air}) = \frac{2||\vec{v}_0||^2 \sin(\theta)\cos(\theta)}{|g|}$$

$$x(t_{air}) = \frac{||\vec{v}_0||^2 \sin(2\theta)}{|g|} \text{ meters}^2.$$

If the launch angle is $\theta = 0.00°$, then $\sin(\theta) = 0$ and $\cos(\theta) = 1$, so the range is

$$x(t_{air}) = ||\vec{v}_0||\sqrt{\frac{2y_0}{|g|}} \text{ meters.}$$

[2] In trigonometry, you likely learned that 2 sin(θ) cos(θ) = sin (2θ). This is a convenient identity that shortens the lengths of many equations and formulas.

Lab 3 Objectives

Projectiles are practical examples of two-dimensional motion. A common question is one involving the two-dimensional motion of an object launched horizontally. In this experiment, predict the horizontal range of a ball launched horizontally from a measured height, a measured initial velocity, and a given value for the acceleration due to gravity.

- Use measured values for launch height and initial velocity to predict the horizontal range of a projectile.
- Establish a foundational understanding of the equations associated with the path of a projectile in two-dimensions.

Materials and Equipment

For each group:

- ☐ PASCO Mini Launcher
- ☐ C-Clamp
- ☐ Large base with metal rod
- ☐ Thumb screws for attaching mini launcher to rod (2)
- ☐ Photogates (1 or 2 per launcher)
- ☐ Photogate Mounting Bracket (2 per group)
- ☐ Digital Adapter with two phone-to-digital cables (1 for each photogate)
- ☐ GLX Data Collection System
- ☐ USB connector for GLX to laptop
- ☐ Laptop running PASCO Capstone software
- ☐ 1.6 cm diameter metal ball
- ☐ Launcher loading rod (plunger)
- ☐ PHYS210_Lab3.cap file
- ☐ Adhesive tape
- ☐ White paper (2 sheets)
- ☐ Carbon paper (1 sheet)
- ☐ Ruler
- ☐ Meter stick
- ☐ Tape measure
- ☐ Board, mat, and bookend to create a block for the launched ball

Safety

In addition to normal laboratory safety expectations, please observe the following additional precautions:

- Do not place foreign objects into the launcher.

$$t_{air} = \sqrt{\frac{2y_0}{|g|}} \text{ seconds}$$

because $\sin(0.00) = 0$.

<u>*Range*</u>

The horizontal *range* of the projectile is the horizontal displacement at time t_{air}:

$$x(t_{air}) = v_{0x} t$$

$$x(t_{air}) = ||\vec{v}_0|| \cos(\theta) \left(\frac{||\vec{v}_0|| \sin(\theta) + \sqrt{||\vec{v}_0||^2 \sin^2(\theta) + 2|g|y_0}}{|g|} \right) \text{ meters.}$$

When $y_0 = 0.00$ meters, this equation simplifies to

$$x(t_{air}) = ||\vec{v}_0|| \cos(\theta) \left(\frac{2||\vec{v}_0|| \sin(\theta)}{|g|} \right)$$

$$x(t_{air}) = \frac{2||\vec{v}_0||^2 \sin(\theta) \cos(\theta)}{|g|}$$

$$x(t_{air}) = \frac{||\vec{v}_0||^2 \sin(2\theta)}{|g|} \text{ meters}^2.$$

If the launch angle is $\theta = 0.00°$, then $\sin(\theta) = 0$ and $\cos(\theta) = 1$, so the range is

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- ☐ Meter stick
- ☐ Tape measure
- ☐ Board, mat, and bookend to create a block for the launched ball

Safety

In addition to normal laboratory safety expectations, please observe the following additional precautions:

- Do not place foreign objects into the launcher.

- Do not look into the launcher.
- Do not use your fingers to push the metal ball into the launcher; always use the plunger.
- Do not aim the launcher at others.
- Always aim the launcher towards the board & mat system at the end of the table.
- Use only what you are provided to engage the ball in the launcher.
- The photogates and adapter are delicate; please use care when attaching and detaching them from the mini launcher.
- Because some of the equipment, particularly the photogates and photogate bracket, are somewhat delicate, use caution and care when attaching these to the mini launcher and when removing cables from the photogates.
- At various points during the instructions, you will see the symbol ⬣. When you see it, stop what you're doing and call your instructor over to your station.

Procedure

Part I: Set Up and General Instructions

At certain points during this laboratory session, you should check with your instructor to make sure that you have attached or detached certain equipment properly and that you are measuring distances correctly. You will see the symbol ⬣ at steps where you will likely wish to consult with your instructor.

The Capstone file for this lab has multiple tabs that are named in an effort to be self explanatory. Ask your instructor if you are unsure if you are on the proper tab for a particular step.

I.1. Two mini launchers are set up for you today. One launches the ball from table height ($y_0 = 0.00$ meters) with a launch angle of 60.0° and the other launches from 0.300 meters (30 cm) above the table with a launch angle of 0.00°.

I.2. The PASCO Capstone software has been set up to recognize the path of the ball through two photogates, positioned 0.100 meters apart and record the velocity by computing

$$||\vec{v}_0|| = \frac{0.100}{t_1} \text{ meters per second}$$

where t_1 is the time, in seconds, it takes for the metal ball to pass through the two photogates. The initial velocity will be estimated by the software.

I.3. Without the photogates attached, verify the height of the launch position using a ruler, meter stick, or tape measure. If there is a discrepancy, ask your instructor to help you adjust the mini launcher. ⬣

I.4. Use the protractor on each mini launcher to verify the launch angle. If there is a discrepancy, ask your instructor to help you adjust the launcher. ⬣

Part II: Launch from 0.00 meters and $\theta = 60°$

II.1. Carefully attach a photogate adapter with one photogate attached to the mini launcher. Make sure that the digital adapter is connected to the GLX in the port labeled as Port 1. The photogate should be positioned closest to the launch point and should be connected to the port labeled 1 on the digital adapter with the phone-to-digital cable. Connect the GLX to the laptop with the USB cable. ⬣

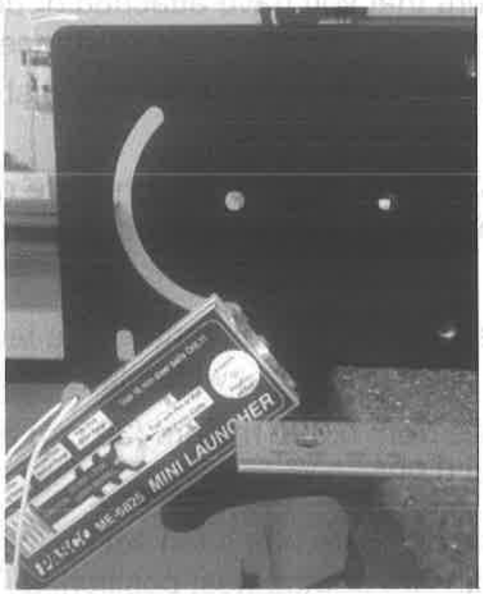

Figure: Mini Launcher Clamped to Lab Bench to Launch from Table Height

II.2. In Capstone, verify that the hardware is set so that the flag length is 0.016 m (the diameter of the ball), and that the spacing between the gates is set to 0.100 meters (the distance between gates). Verify that the software will capture the velocity through the photogate. Open today's experiment file.

II.3. Place the metal ball into the launcher and use the plunger to push it into the launcher until you hear one click (this sets the launcher to *short range*).

II.4. Begin recording on Capstone.

II.5. Pull up on the yellow cord to launch the ball and check to see that (a) the software reports the initial velocity it estimated and (b) that the board and mat are situated to block the ball from rolling off the table. Ask your instructor for assistance before proceeding if no velocity is recorded or if the ball is not properly blocked. Do not stop recording on Capstone.

II.6. Disconnect the cable from the photogate, and re-insert the ball.

II.7. Reconnect the cable to the photogate and launch the ball a second time. (You are asked to disconnect and reconnect so that neither your hand, arm, or the plunger activates the light in the photogate).

II.8. Repeat until you have 3 visible values for the initial velocity. Report the mean of the three velocities here:

$||\vec{v}_0|| \approx$ __________________ meters per second

II.9. Stop recording on Capstone.

II.10. Open the calculator in Capstone and change the value of *velocityAt60Degrees* to your estimated velocity. Close the calculator.

II.11. Use the formula

$$t_{air} = \frac{2||\vec{v}_0||\sin(\theta)}{|g|} \text{ seconds}$$

to estimate the air time of the ball. Record the value below:

$$t_{air} = \frac{2\|\vec{v}_0\| \sin(\theta)}{|g|} \approx \_\_\_\_\_\_\_\_\_\_\_\_ \text{ seconds}$$

II.12. Estimate the *range* of the projectile using the formula

$$x\left(t_{air}\right) = \frac{\|\vec{v}_0\|^2 \sin(2\theta)}{|g|} \approx \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \text{ meters}$$

II.13. Remove the photogate bracket from the mini launcher and carefully set it aside, out of the way of the path of the ball. ⬢

II.14. Use the meter stick (or tape measure) to locate the position on the table that is $x\left(t_{air}\right)$ meters from the launch point.

II.15. Center one piece of white paper over this point on the table and use tape to secure it.

II.16. Place a piece of carbon paper (shiny side down) over the white paper and secure it with additional tape.

II.17. Launch the ball and make sure that it hits the paper. Check your calculations if it does not or ask your instructor for assistance. Reposition the papers as necessary so that the ball does land on the paper. ⬢

II.18. Launch the ball until it has hit the paper 5 times.

II.19. Carefully remove the carbon paper and leave the white paper securely in place on the table.

II.20. Use the meter stick or tape measure to measure the *horizontal* distance from the launch point to each of the 5 dots on the paper. (Do not tilt the meter stick at an angle.) ⬢

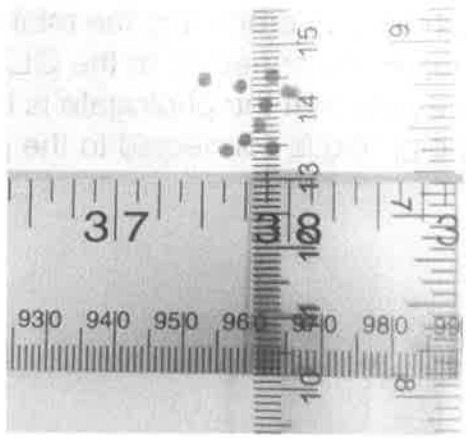

Figure: Meter Stick with Clear Ruler as Guide for Measuring Distance

II.21. Record each distance in Table 1 below:

Table 1: Range Measurements for $y_0 = 0.00$ meters

Position	Distance (meters)
1	
2	
3	
4	
5	
Mean:	

II.22. Compute the *percent error* for the range by comparing the mean obtained in Table 1 with the value calculated in II.11. Use three significant figures.

$$\text{Percent Error} = \frac{|Mean - Calculated|}{Calculated} \times 100\% = \_\_\_\_\_\_\_\_\_\_\%$$

II.23.

Part III: Launch from 0.300 meters and $\theta = 0.00°$

NOTE: Do not push down on the launcher apparatus or you could affect the height of the launch or damage the launcher itself.

III.1. Carefully attach the photogate bracket to the mini launcher. Make sure that the digital adapter is connected to the GLX in the port labeled as Port 1. Make sure that the photogate is in the position closest to the launch point and is connected to the port labeled 1 on the digital adapter. ●

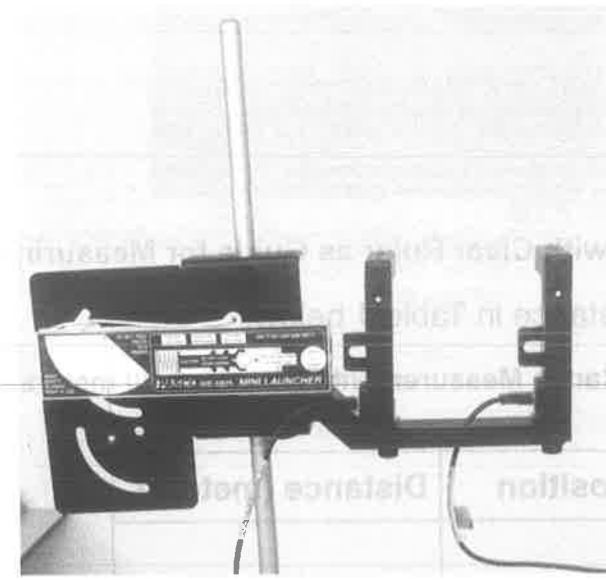

Figure: Mini Launcher with Two Photogates, Attached to a Rod on a Large Base with a Thumb Screw at the Top Position

III.2. In Capstone, verify that the hardware is set so that the flag length is 0.016 m (the diameter of the ball), and that the spacing between the gates is set to 0.100 meters (the distance between gates). Verify that the software will capture the velocity through the gate.

III.3. Place the metal ball into the launcher and use the plunger to push it into the launcher until you hear one click (this sets the launcher to *short range*).

III.4. Begin recording on Capstone.

III.5. Pull up on the yellow cord to launch the ball and check to see that (a) the software reports the initial velocity it estimated and (b) that the board and mat are situated to block the ball from rolling off the table. Ask your instructor for assistance before proceeding if no velocity is recorded or if the ball is not properly blocked. Do not stop recording on Capstone. ●

III.6. Disconnect the cables from the photogates only, and re-insert the ball.

III.7. Reconnect the cables to the photogates and launch the ball a second time.

III.8. Repeat until you have 3 visible values for the initial velocity. Report the mean of the three velocities here:

$||\vec{v}_0|| \approx$ ____________ meters per second

III.9. Stop recording on Capstone.

III.10. Open the calculator and change the value of *velocityAt0Degrees* to your estimated velocity. Close the calculator.

III.11. Use the formula

$$t_{air} = \sqrt{\frac{2y_0}{|g|}} \text{ seconds}$$

to estimate the air time of the ball. Record the value below:

$$t_{air} = \sqrt{\frac{2y_0}{|g|}} \approx \text{\_\_\_\_\_\_\_\_\_\_\_\_ seconds}$$

III.12. Estimate the *range* of the projectile using the formula

$$x(t_{air}) = ||\vec{v}_0||\sqrt{\frac{2y_0}{|g|}} \approx \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ meters}$$

III.13. Remove the photogate adapter from the mini launcher and carefully set it aside, out of the way of the path of the ball.

III.14. Attach the spare thumbscrew to the lower position of the launcher base and tighten it so that it is flush with the metal rod and

III.15. Use the meter stick (or tape measure) to locate the position on the table that is $x(t_{air})$ meters from the launch point.

III.16. Center one piece of white paper over this point on the table and use tape to secure it. Do not use the same paper as you used in Part I.

III.17. Place a piece of carbon paper (shiny side down) over the white paper and secure it with additional tape.

III.18. Launch the ball and make sure that it hits the paper. Check your calculations if it does not or ask your instructor for assistance. Reposition the papers as necessary so that the ball does land on the paper.

III.19. Launch the ball until it has hit the paper 5 times.

III.20. Carefully remove the carbon paper and leave the white paper securely in place on the table.

III.21. Use the meter stick or tape measure to measure the *horizontal* distance from the launch point to each of the 5 dots on the paper. (Do not tilt the meter stick at an angle.)

III.22. Record each distance in Table 1 below:

Table 2: Range Measurements for $y_0 = 0.00$ meters

Position	**Distance (meters)**
1	
2	
3	
4	
5	
Mean:	

III.23. Compute the *percent error* for the range by comparing the mean obtained in Table 2 with the value calculated in III.12. Use three significant figures.

Percent Error = $\frac{|Mean - Calculated|}{Calculated} \times 100\%$ = ________

Analysis

1. Much of the analysis for this experiment was completed via the calculations and percent error in each part. Make sure that you include the equations, means, and percent errors for each part of the experiment in your discussion.
2. Make sure that you explain, in your own words, why we were able to assume that horizontal acceleration is 0.00 meters per second squared. (Hint: There is no horizontal acceleration, just $\vec{g}$, acting downward on the ball once it has launched. If you need additional instruction or guidance on this topic, see the lesson on two-dimensional motion at Khan Academy.)

Discussion

Your discussion section must include the answers to the following questions. Try to limit your discussion section to two paragraphs.

1. We discussed the formulas, in general, in class. Did this experiment help you to understand the formulas for airtime and horizontal range? What else have you learned about motion in two dimensions from this experiment? Does the mass of the ball come into play? Why or why not? (Hint: Look at the formulas for y(t) and x(t).)
2. Since acceleration is constant, vertical, and negative, does it make sense that a projectile follows a parabolic path when no horizontal forces are acting on it? (This last question relates to the calculus derivation I did, starting with acceleration and integrating to get velocity, then integrating to get position in both the x- and y-directions.)
3. The velocity of the projectile in the horizontal direction is constant. Is the vertical velocity increasing, decreasing, or constant? Why? (Note: To help you answer this question, remember that, for negative numbers, the larger the magnitude, the smaller the number.)
4. What errors did you encounter in the experiment? Use the "Types of Errors" list in the background for Laboratory 1 to help you to determine the source of errors.