Energy

The Work-Energy Theorem and Conservation of Energy

Background

We have seen several forces acting on an object and observed Newton's Laws of Motion experimentally. We have also considered various forces acting upon moving objects. Now it is time to tie these together with the Work-Energy Theorem

Work

Work is the *energy* transferred from one object to another when forces are applied to the object over a distance or *displacement*. **Work** is done parallel to displacement and is a vector quantity. The relationship between **work** and **force** is linear:

$$\vec{W} = \vec{F}\,\Delta x,$$

where $\vec{F}$ is measured in Newtons and Δx is measured in meters. In this case, the applied force is in the same direction as displacement. The unit of work is sometimes referred to as a *Newton-meter* or a *Joule* (J). In basic units, work is represented by $\frac{kg \cdot m^2}{s^2}$.

The work done by a force acting on an object is $\vec{0}$ Joules if there is no motion. Similarly, the work done is also $\vec{0}$ Joules if the net force acting on the object is perpendicular to the displacement of the object.

When multiple forces are acting on an object, the net work done is

$$\vec{W}_{net} = \sum_{i=1}^{n} \vec{F}_i \Delta x_i \text{ Joules}$$

Energy

Energy is defined to be the capacity to do **work**. This is somewhat of a circular definition, since work is about transferring energy between objects and energy is a measure of the ability for that transfer to occur. There are two important, relevant types of **energy—potential energy** and **kinetic energy**.

Gravitational Potential Energy

Gravitational potential energy is based on the position and mass of an object; it is a concept that quantifies stored energy of a static object that has the potential to do work if set into motion. The most general equation for **gravitational potential energy** is

$$PE = \vec{F}_{gravity} h$$

$$PE = m \vec{g} h.$$

where *m* is the mass of the object in kg; $\vec{g} = -9.81 \vec{j}$ m/s^2; and *h* is the height of the object, with respect to a horizontal surface, measured in meters. Thus, the units on **potential energy** are

$$kg \cdot \frac{m}{s^2} \cdot m = \frac{kg \cdot m^2}{s^2} = Newton \cdot meters = Joules.$$

Kinetic energy, on the other hand, is derived from the mass of the object and its velocity. It represents the energy required to move an object. The general equation for **kinetic energy** is found by combining Newton's Second Law of Motion ($\vec{F} = m \vec{a}$) and the kinematic equation $v^2 = v_i^2 + 2\|\vec{a}\| \Delta x$.

First, we look at the magnitudes of the vectors in Newton's Law, then solve for the magnitude of acceleration:

$$\|\vec{F}\| = m \|\vec{a}\|$$

$$\|\vec{a}\| = \frac{\|\vec{F}\|}{m}.$$

Next, we solve for the magnitude of acceleration in the kinematic equation:

$$\left\|\vec{a}\right\| = \frac{1}{2}\left(\frac{v^2 - v_i^2}{\Delta x}\right).$$

Equating the two resulting expressions for the magnitude of acceleration, we have:

$$\frac{\left\|\vec{F}\right\|}{m} = \frac{1}{2}\left(\frac{v^2 - v_i^2}{\Delta x}\right)$$

$$\frac{m \cdot |g|}{m} = \frac{1}{2}\left(\frac{v^2 - v_i^2}{\Delta x}\right)$$

$$m\,|g|\,\Delta x = \frac{1}{2}m\left(v^2 - v_i^2\right)$$

On the left-hand side of this equation, we have the product of the mass of an object, the magnitude of the acceleration due to gravity, and displacement in the direction of motion. That's the magnitude of the *force due to gravity* times displacement. It's **work!**

On the right-hand side, we have one-half the mass of the object times the square of its velocity minus the square of its initial velocity.

The consequence of the equation is called the **work-energy theorem.**

The two sides of the equation together tell us that the work done in the direction of motion is related to the velocity of the object and its mass. Thus, the object in motion is doing work. The amount of work a moving object can do is therefore its **kinetic energy** and is

$$KE = \frac{1}{2}m\left(v^2 - v_i^2\right).$$

You can quickly verify that the units here are Joules. When the initial speed is 0 meters per second, we write

$$KE = \frac{1}{2}m\,v^2$$

where $v = \left\|\vec{v}\right\|$ is the speed at the moment that total displacement is Δx.

Now it should be clear why **work** and **energy** have circular definitions—they have the same dimensions (units).

There is another way to find **kinetic energy** if we consider an object with mass *m* starting at rest (no velocity) and moving horizontally with speed v in the horizontal direction. The object's capacity to do work over a very small change in its position is (again) related to Newton's Second Law of Motion:

$$\Delta W = m\, a\, \Delta x \approx m \frac{\Delta v}{\Delta t} \Delta x = m \frac{\Delta x}{\Delta t} \Delta v$$

Since $v = \frac{dx}{dt} \approx \frac{\Delta x}{\Delta t}$ for very short intervals of time, the total capacity to do work over a total displacement $\Delta x_1 + \Delta x_2 + \cdots + \Delta x_n$ is

$$W_{net} = \sum_{i=1}^{n} \left(m \frac{\Delta x}{\Delta t} \Delta v \right) = \sum_{i=1}^{n} (m\, v\, \Delta v)$$

As the intervals of time decrease ($\Delta t \to 0$), the Fundamental Theorem of Calculus gives us the following equation:

$$W_{net} = \int_{v_i}^{v_f} (m\, v)\, dv = \left[\frac{1}{2} \left(m\, v^2 \right) \right]_{v_i}^{v_f} = \frac{1}{2} m \left(v_f^2 - v_i^2 \right),$$

where, as before, v_i is the initial speed and v_f is the final speed of the object over the total displacement.

The Work-Energy Theorem

The **Work-Energy Theorem** is a formula we have already derived:

$$W_{net} = \Delta KE = KE_f - KE_i = \frac{1}{2} m\, v_f^2 - \frac{1}{2} m\, v_i^2,$$

where the subscript *f* indicates final time and the subscript *i* represents initial time. Paraphrasing, the theorem states that the net work done by all forces acting on an object is equal to the change in kinetic energy from start to finish.

You might wonder why PE is nowhere to be seen in the Work-Energy Theorem. Don't worry; it's in there, hiding inside the net work done by all forces.

Work and Energy Along an Inclined Plane

Now we will consider the situation of an object moving along an inclined plane (Figure 1). There are many preliminary equations to consider.

A mass of m kilograms on an inclined plane has gravitational potential energy of $m\,|g|\,h$ Joules, where h is its vertical height from a horizontal reference, which in our case will be the lab bench. Once set in motion, the object will have, at different locations along its path, kinetic energy of $\frac{1}{2} m\, v^2$ Joules, where v is the speed of the object at a location Δx.

If the object has been displace Δx meters, its capacity to do work is its **total energy**, $E = PE + KE$. All along its path, the object's **total energy** will not change because **energy is conserved** due to the **Law of Conservation of Energy**. Energy can neither be created nor destroyed, but can be transferred between objects or transformed into another form of energy.

For an object that starts at rest, if we let *x* be the position, in meters, of the object and *v* represent its speed, we have the following equations for the object at any time *t* during its motion:

- $h = x\sin(\theta) = x\frac{H}{L}$ meters
- $PE = m|g|h$ Joules
- $KE = \frac{1}{2}mv^2$ Joules
- $E = PE + KE = m|g|h + \frac{1}{2}mv^2$ Joules

Overview of the Experiment

As before, when working with an inclined plane, we will set our reference frame so that the *x*-axis is parallel to the inclined plane (Figure 1).

Since our experiment will only involve displacement Δx, we'll focus our attention on components related to the $-\vec{\imath}$ and $\vec{\imath}$ directions and drop the vector notation since we'll only be focused on one axis.

We will set the cart at the bottom of the plane and push it upwards, allowing it to roll freely back down the plane and stop its motion. In doing so, we will see the relationships between position, velocity, kinetic energy, and potential energy, as well as visualize the Law of Conservation of Mechanical Energy.

At the bottom of the plane, when the object is at rest, it has no kinetic energy. If force is applied to move the object up the plane, then it will have both potential and kinetic energy at any instant of time *t* seconds.

Gravitational potential energy will increase as the object travels up the plane, since the height of the object will increase.

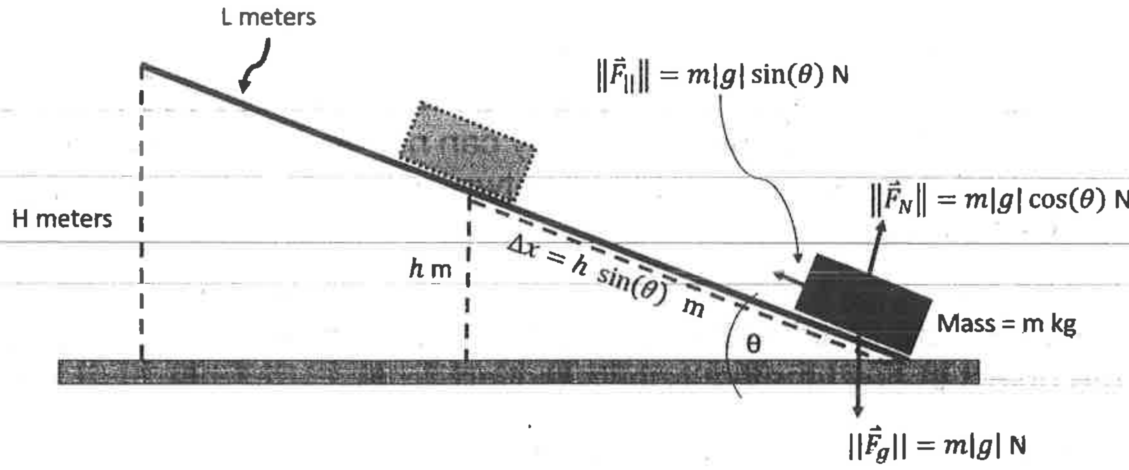

Figure 1: Diagram Demonstrating Forces, Position, and Motion of an Object on an Inclined Plane

Similarly, its kinetic energy will decrease because the object will slow down as it moves higher because the acceleration due to gravity will overcome the acceleration resulting from the original applied force.

The potential energy will decrease as the object moves back down the plane, and its kinetic energy will increase because its velocity will increase.

Laboratory Objectives

A cart with mass *m* is pushed upwards along an inclined plane and allowed to freely roll until it reaches the bottom of the plane again. During and after this laboratory session, you will

- Observe position and velocity graphs and identify key times where the graphs exhibit important behavior (maximums and minimums);
- Compute gravitational potential energy and kinetic energy instantaneously along the path of the cart, noting important changes in the graphs of both versus time;
- Visually verify the Law of Conservation of Energy for this mechanical system and identify possible reasons why the Law *appears* to fail at certain times during the cart's motion.

Materials and Equipment

- Pasco Smart Cart with rubber bumper attachment
- 250 g cart masses (2)
- Balance

- Frictionless dynamics track
- Large base and support rod
- Track rod clamp
- Meter stick
- Track end stop (1)
- Elastic bumper brackets (2)
- Elastic bumper material (3 pieces, 10 cm in length)
- Laptop
- Pasco Capstone software
- PHYS_Lab7.cap file

Safety

Normal laboratory safety features should be followed while completing this experiment. Be careful not to let the dynamics cart fall off of its track, crash into the end stop, crash through the elastic bumper, or fall onto the floor.

The cart masses are heavy; do not drop them onto any equipment or onto the floor as they will cause damage.

Procedure

Please pay careful attention to the steps given below and listen to your instructor when she explains the experimental design for today.

As always, some steps were completed for you before you entered the lab.

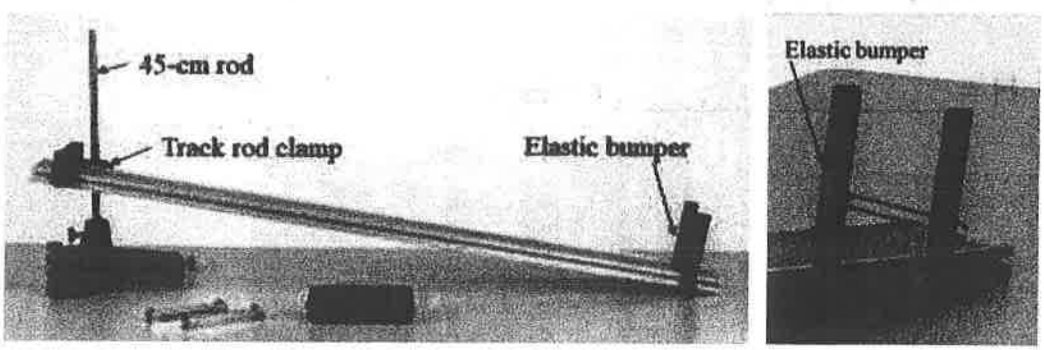

Figure 2: Laboratory 7 Set Up

Part I: Set Up

I.1. Attach the track rod clamp to the support rod.

I.2. Remove the feet, if necessary, from the dynamics track.

I.3. Slide the track onto the clamp and push it up about 5 cm past the support rod. Tighten the clamp and make sure that the track is stable and level.

I.4. Adjust the height of the clamp on the support rod to make sure that the track is approximately 20 cm above the table.

I.5. Attach the end stop to the top of the track, with the magnetic side pointing in the opposite direction as the track.

I.6. Attach the elastic bumper brackets to each side of the end of the track. Do not over tighten the thumb screws, but make sure that the brackets are vertical.

I.7. Pull one piece of the elastic material through the slots in each bracket and push it into the slots in each bracket; repeat for the other two pieces of material to form an elastic barrier.

I.8. With the meter stick, measure the height of the track at the top, from the table to the underside of the track (Figure 3). Record the height, in meters, below:

H = ________________ meters

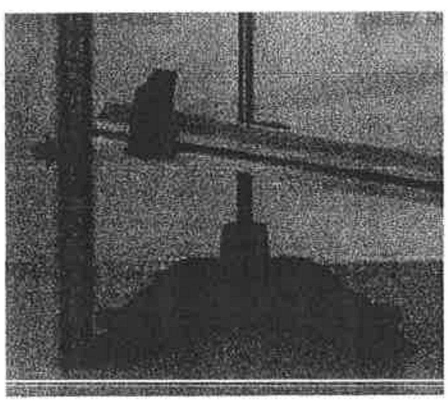

Figure 3: Measuring the Height of the Track

I.9. Slide the meter stick onto the track so that its 0 end is flush with the top of the track and the opposite end of the meter stick is under the elastic bands on the bumper. Measure the length of the track, in meters, and record it below:

L = ______________ meters

I.10. Turn on the laptop and load the file for today's lab.

I.11. Turn on the Smart Cart, if necessary. You'll know that the cart is "on" if its LED lights are blinking.

I.12. Click on the "Hardware Setup" button on the left of Capstone and connect to the Smart Cart you were given. Each Smart Cart is labeled with an identifying six digit number; make sure you connect to the correct cart. Double click on the name of your Smart Cart to connect.

I.13. Disable all but the "Smart Cart Position Sensor" (Figure 3). Note that this and the previous step cannot be done without Capstone software running.

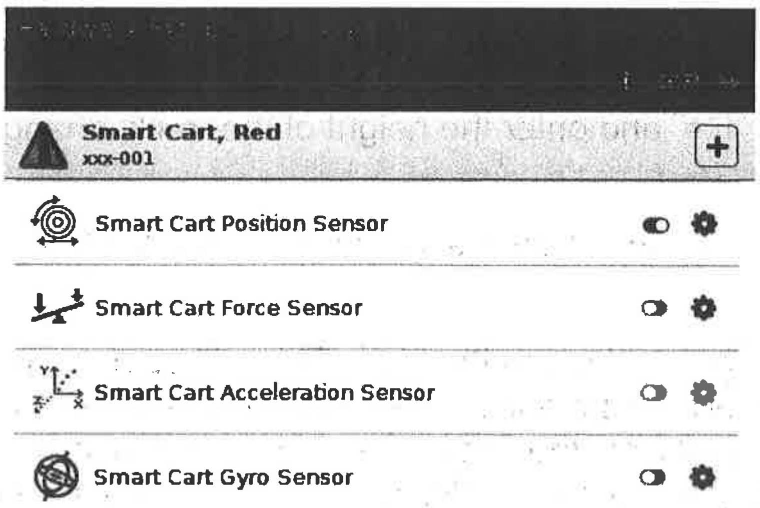

Figure 3: Setting Up the Smart Cart

I.14. Close the Hardware Setup screen by clicking its button again.

I.15. On the right, drag the "Graph" icon  onto the Capstone display screen.

I.16. Click the  button to add a second graph to the display.

I.17. Click on "Select Measurement" on the bottom of the screen and choose time.

I.18. On the top graph, click on the "Select Measurement" button on the vertical axis and choose position.

I.19. On the lower graph, select velocity as the displayed measurement.

I.20. Change the Sample Rate at the bottom of the screen to 50 Hz.

I.21. Place the cart upside down on the balance and place both cart masses on top of it. Measure the total mass, in kilograms, and record it below:

m = ______________ kg

I.22. Open the Calculator in Capstone. You will enter constants and formulas in the next several steps, each on a different line in the Calculator screen.

I.23. Type "m=" and enter the mass; change the units to "kg".

I.24. Type "H=" and enter the height of the track; change the units to "m".

I.25. Type "L=" and enter the length of the track; change the units to "m".

I.26. Type "h=" and press the [key. This will open a menu with user-entered expressions and constants, as well as values that can be obtained from the Smart Cart. Choose "Position (m)". Type *, and press [again; choose "Constants" and "H". Type /, then [, and choose "L" from the list of "Constants". Change the units to "m".

I.27. Type "PE=" and press [again. Choose "m" from the "Constants" menu, press * and then [, and choose "Acceleration due to gravity" from the "Constants" menu. Finally, press [and select "h" from the "Constants" menu. Change the units to "J".

I.28. Type "KE=0.5*" and press [to open the menu; choose "m" from the "Constants" menu and press * again, followed by [. Now choose "Velocity (m/s)" and type ^2. Set the units to "J".

I.29. Finally, type “E=”, press [and choose “PE” from the list of equations. Type “+” and press [and choose “KE” from the list of equations. Change the units to “J”.

I.30. Check that your calculator screen looks similar to Figure 4. Your values for m, H, and L will likely be different.

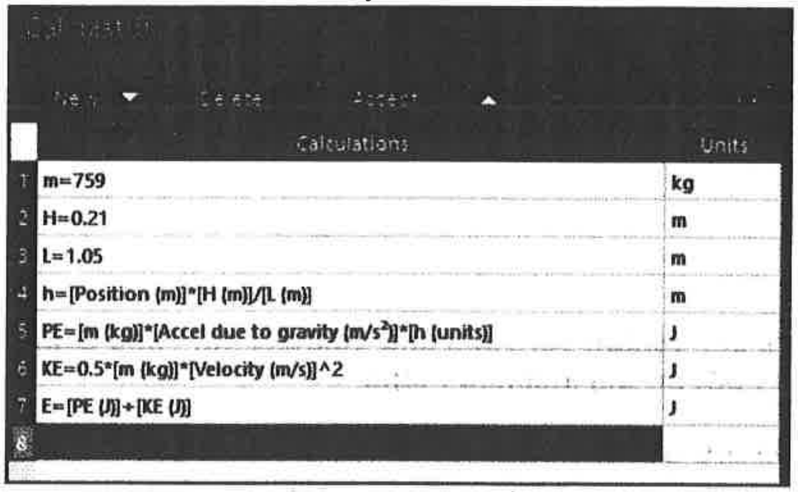

Figure 4: Capstone Calculator Screen

I.31. Close the calculator screen.

Part II: Data Collection

II.1. Carefully set the cart onto the lower end of the track, making sure that its wheels are in the track slots. Also make sure that the symbol “+x→” is pointing towards the top of the track.

II.2. Place the cart masses side-by-side in the cart. They should be flat on the cart.

II.3. Press the record button and push the cart up the track with enough force that it *nearly* reaches the top but does not hit the end stop. If the cart hits the end stop, cease recording and delete the run.

II.4. Allow the cart to roll back down the track and simultaneously stop it with one hand while stopping the recording.

II.5. Your Position vs. Time and Velocity vs. Time graphs should be similar to Figure 5. If not, delete the run and try again

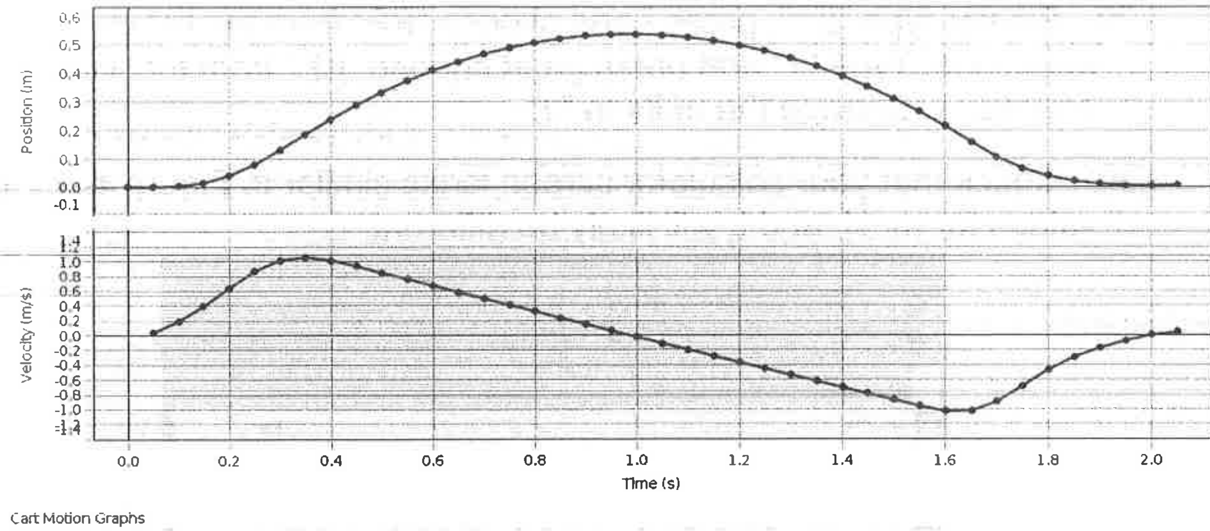

Figure 5: Position vs. Time and Velocity vs. Time Graphs for the Loaded Carts Path on the Inclined Plane.

II.6. There are several moments in time that are important for understanding the relationships between displacement (position), speed (the absolute value of velocity), potential energy, kinetic energy, and total energy.

a. Press the  button at the top of the graphics display and choose the multi-coordinate option. A movable bar will appear; slide the bar to the left, coinciding with the point just after you began to push the cart.

b. Repeat and move the bar to the point where the velocity is at its maximum. This corresponds to the moment when the cart begins to move up the track on its own.

c. Repeat and move the bar to the point where the position is at its maximum. This corresponds to the point where the cart's velocity was 0 m/s and began to move back down the track.

d. Repeat and move the bar to the point where the velocity is at its minimum (fastest downward velocity). This corresponds to the moment when the cart is freely moving down the track, before you stopped it.

e. Repeat again and more the bar to the point just before the cart came to rest.

II.7. Save your graph to a Google Doc and share it with your lab partners.

II.8. Complete Table 1 with the values (3 significant figures) obtained from the coordinates on the graphs.

Table 1: Data from the Cart Motion Graphs

Time (s)	Position (m)	Velocity (m/s)	Description

II.9. Add a second worksheet and drag the "Graph" icon onto the display.

II.10. Press the twice to create a display with three graphs. Select time for the horizontal axis. On the top graph, choose "PE" for the vertical axis. On the middle graph, choose "KE" for the vertical axis. Choose "E" for the vertical axis on the bottom graph. Your graph should be similar to Figure 6.

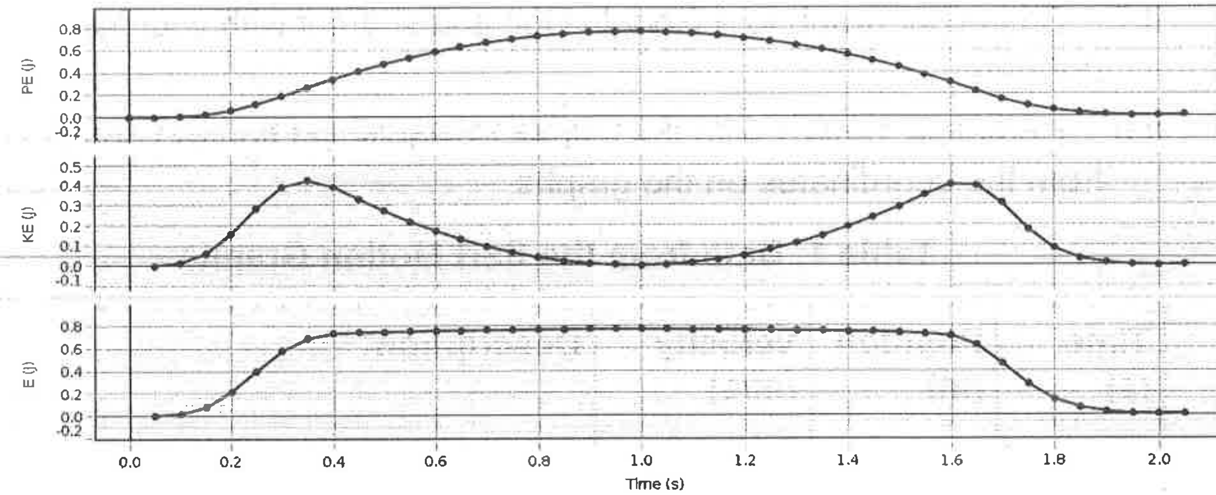

Figure 6: Potential Energy, Kinetic Energy, and Total Energy for a Loaded Cart Traveling on an Inclined Plane

II.11. Using the multi-coordinate tool, place bars at the same times at which you placed them on the position and velocity graphs. Record the values obtained in Table 2.

II.12. Save your graph in the same Google Doc.

Table 2: Kinetic, Potential, and Total Energy

Time (s)	KE (J)	PE (J)	E (J)	Description

Discussion

Please use paragraphs to conclude your report. You must include both graphs in your lab report. There is no analysis or error analysis for this lab. The conclusion needs to include but is not limited to:

1. What is the relationship between **work** and **energy**?
2. What relationships did you notice between the maximum and minimum velocities and the kinetic energy at the corresponding times?
3. What relationship did you notice between the maximum position and the potential and kinetic energies at the same time?
4. During a specific time interval, the total energy E should appear to be constant. This aligns with the Law of Conservation of Energy. What time interval supports this law?
5. Before the time interval mentioned above, it appears that energy has not been conserved. In fact, there is a point where total energy, kinetic energy, and potential energy are all 0 Joules. This is because your hand was holding the cart, keeping it from moving; since the data in the graphs comes only from the cart, where was all the energy at that instant?
6. Briefly, it appears that the total energy is increasing. Energy was being transferred *to* the cart during this time interval. From what was the energy being transferred?
7. Later, the total energy appears to be decreasing. However, energy was being transferred *from* the cart during this particular interval. To what was the energy being transferred?
8. As the cart moves *up* the track, describe the behavior of the potential, kinetic, and total energies. Are they increasing? Decreasing? Constant? When? Why?
9. As the cart moves *down* the track, describe the behavior of the potential, kinetic, and total energies. Are they increasing? Decreasing? Constant? When? Why?